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NONLINEAR ROTARY WAVE ION PLASMA

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Bernard H. Duane and Harold B. Liemohn

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TECHNICAL REPORT SUMMARY

The geomagnetic cavity surrounding the earth, called the magnetosphere, contains an enormous amount of energy in the form of electromagnetic waves, trapped energetic particles, and the static magnetic field. The waves and particles are in a state of dynamic equilibrium as they interact with each other through various plasma instabilities. Our understanding of this equilibrium state has progressed to the point where it may be possible to stimulate wave amplification at ULF ($\sim$0.1-10 Hz) and VLF ($\sim$1-100 kHz) by disturbing the equilibrium through catalytic injections of plasma clouds or particle beams. Such a prospect has great significance. First, it offers a direct test of the fundamental physical processes that are believed to control the state of the magnetosphere. Second, controlled injections might be used to stimulate artificial amplification of VLF or ULF emissions on command and allow them to be used as a wide-area communication system.

One of the dominant plasma instabilities in the magnetosphere is the cyclotron-resonance interaction between energetic particles and electromagnetic waves. The condition for the interaction to occur is that the motion of the particle Doppler shifts the wave frequency to its local cyclotron frequency. The net exchange of energy between a band of waves and a group of particles depends upon the shape of the phase-space distribution of the particles and the local frequency parameters of the plasma medium. The interaction is undoubtedly responsible for significant wave amplification and particle precipitation in certain circumstances and appears to be the process responsible for maintenance of an equilibrium configuration of waves and particles in the magnetosphere.

These qualities make this interaction a logical first choice for investigation as a means of stimulating artificial amplification of VLF and ULF waves in the magnetosphere. The interaction is sensitive to both the background magnetosphere parameters and the hot energetic particle distribution. Modification of the interaction can be achieved by the introduction of localized clouds of plasma that reduce the phase velocity of the waves and cause enhanced amplification by lowering the resonance velocity. Alternatively, direct injection of hot energetic particle beams can also appreciably alter the local amplification characteristics.

Previous studies of stimulated amplification have been based on the linear perturbation theory for small wave amplitudes. Their results have indicated promising methods for generation of signals, but the perturbation theory breaks down for realistic amplification rates. Thus the nonlinear solution of the problem is essential for a quantitative description of useful power levels for stimulated emissions. However, the nonlinear aspects of the interaction have been particularly elusive because its exact solution is described by elliptic functions in the complex plane. This fact has prevented useful application of the results because many side constraints and most constants of the motion remained undiscovered.

This year, really significant theoretical progress has been achieved at BNW on the exact nonlinear problem. Foremost was the identification of five new constants of the motion. These constants, combined with two previously known constants, complete the set that is required to fully specify the behavior of the interaction. An entire new vista of applications is now accessible. For example, instead of integrating the plasma kinetic equations point by point in the six-dimensional phase space, it is possible to write formal solutions for the particle distribution that convert the equations into a straightforward eigenvalue problem. (Imagine integrating the Schrödinger wave equation over all momentum and configuration space to ascertain the energy levels of hydrogen, as opposed to solving the much simpler eigenvalue problem.) Of course, the solution of the eigenvalue problem in the complex plane is by no means trivial, particularly for realistic conditions in the magnetosphere, but a vital step has been accomplished in understanding the physical process and evaluating the nonlinear growth rate.

The mathematical technique for the nonlinear solution is fully established in the accompanying document. A computer program was also prepared to evaluate the closed form solution. The program includes subroutines that probe all aspects of the interaction: particle trajectories, arbitrary phase space distributions, nonlinear wave amplitudes, and systematic convergence in the eigenvalue hyperspace. The program contains several alternate computing, printing, and plotting subroutines. Unfortunately support for the research was exhausted before the program could be fully debugged and no numerical output is available for magnetospheric applications. (A few numerical solutions have been obtained for waves in fusion plasmas.)

The theory is limited at present to propagation in uniform homogeneous media. For magnetospheric applications the medium is slowly varying so that numerical results from this new nonlinear solution would be helpful in the study of stimulated amplification. More significantly, the present computer program is the essential core for an outer loop that links a series of cells which approximate nonuniform and inhomogeneous media.

In conclusion our nonlinear solution provides the proper machinery for a quantitative assessment of various techniques for stimulation of ULF and VLF emissions in the magnetosphere. We believe our study should be continued to bridge the gap between the esoteric mathematical expressions which are now available and quantitative evaluation of specific plasma injection experiments in the magnetosphere.

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The research work on this ONR Contract to Battelle-Northwest was performed in part by BNW staff and in part by Aerospace Corp staff through a subcontract. The information in this report describes only the BNW research which involved $45,000 of the program. The remaining $40,000, less fee, supported the Aerospace research which is reported in the document, "Wave-Particle Interactions in the Magnetospheric Plasma." [ATR-74(7420)-1] compiled by M. Schulz.

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ABSTRACT

The interaction between a rotary electromagnetic plane wave propagating along an axial magnetic field and an arbitrary phase-space distribution of charged particles is solved exactly for selected cases. The interaction is described by the nonlinearly coupled Vlasov-Maxwell ion-plasma field equations which are integrated with the assistance of four new constants of the motion. The ion orbits in momentum space are bipolar doubly periodic eigenfunctions of ion proper time, which are obtained in closed form as the difference between two doubly quasi-periodic Weierstrass zeta functions. The ion orbits in position space are helical-spiral doubly quasi-periodic functions of ion proper time, expressible simply in terms of doubly quasi-periodic Weierstrass sigma functions. The complete ion distributions are flexible functions of six constants of the ion motion: wave-frame ion energy, transverse gyro center, an inner Hamiltonian correlating wave-frame ion momentum with wave-frame axial position, and both first and second axial integration constants. The eigenvalue determination intricately interrelates the wave propagation vector, the wave amplitude, the axial magnetic field, the double periods, and the bipole separation. The solutions have potential application to stimulated amplification of electromagnetic noise in space, induced particle precipitation at the base of the magnetosphere, and radiative absorption and emission processes in fusion plasmas.

TOPICS

INTRODUCTION

The exchange of energy between electromagnetic waves and magnetically confined particles via the cyclotron-resonance interaction has been the object of intensive study for many years[1-4] due to its applicability in space physics[5-11] and plasma fusion research.[12-13] The wave-particle interaction is described by the coupled Vlasov-Maxwell equations which have well-known perturbation solutions for small wave amplitudes and energy exchange rates.[1-8] However, in many applications, such as stimulated wave amplification and particle precipitation in the magnetosphere[14-18] and radiative losses from fusion machines,[12-13] the perturbation assumptions are no longer valid, and the exact nonlinear solution is mandatory.

Charged particle orbits under the influence of a uniform magnetic field and circularly polarized electromagnetic wave are known to be described by complex elliptic integrals.[19] In the wave frame, the particle kinetic energy and an inner Hamiltonian were well established as constants of the motion,[20] but four other constants remained unknown. Nevertheless, partial integration of the exact equations for the wave-particle interaction was achieved using the available orbit properties.[21,22]

In our analysis, all six constants of the particle motion are derived and physically identified. With these relations it is possible to formulate a flexible phase-space distribution function which satisfies the Vlasov equation. Furthermore, these constants also permit explicit integration of Maxwell's vector wave equation which reduces to a nontrivial eigenvalue problem involving a multidimensional complex hyperspace. The method for numerical evaluation of the proper set of eigenvalues is described and exact solutions are given for two and three component plasmas.

PLASMA PHYSICS

In the four-dimensional space-time of special relativity spanned by the line element

$$d\delta^2 = dx^2 + dy^2 + dz^2 + dict^2 = -c^2 d\tau^2,$$

the nonlinearly coupled Vlasov and Maxwell ion-plasma field equations, relating the electromagnetic vector potential $\underline{A}(\underline{R})$ and the probability density $n_s(\underline{R},\underline{P})$ for ions of charge q_s and rest mass m_s at four-position $\underline{R} = (x\ y\ z\ ict)$ with four-momentum $\underline{P} = m_s d\underline{R}/d\tau = (p_x\ p_y\ p_z\ iE/c)$, take the form[4,13,23-27]

$$\boxed{\begin{aligned} &[\partial n_s(\underline{R},\underline{P})/\partial\underline{R}]\cdot\underline{P}/m_s + [\partial n_s(\underline{R},\underline{P})/\partial\underline{P}]\cdot[q_s\nabla\times\underline{A}(\underline{R})]\cdot\underline{P}/m_s = 0, && (1)\\ &-\nabla\cdot[\nabla\times\underline{A}(\underline{R})] = \mu\textstyle\sum_s\int(q_s\underline{P}/m_s)n_s(\underline{R},\underline{P})dV(\underline{P}), && (2)\end{aligned}}$$

where c is the velocity of light, τ is ion proper time, E is ion energy, $\nabla\times\underline{A}$ is the tensor curl of $\underline{A}$, μ is the magnetic permeability of free space (in rationalized mks units), and the last term sums the four-vector current-and-charge sources over all ion species s and their four-momentum volume elements $dV(\underline{P})$.

The least intricate known method of generating exact solutions to these coupled equations proceeds through two formidable steps:

Step I. Postulation of an analytically tractable functional form for the electromagnetic vector potential $\underline{A}(\underline{R})$, followed by explicit integration of the corresponding orbit equations for each ion species,

$$d\underline{R}(\tau)/d\tau = \underline{P}(\tau)/m_s, \tag{3}$$

$$d\underline{P}(\tau)/d\tau = q_s[\nabla\times\underline{A}(\underline{R})]\cdot\underline{P}(\tau)/m_s. \tag{4}$$

Step II. Assembly of the corresponding set of solutions of the Vlasov equation (1) as a flexible function of the constants of the motion for the ion orbits, followed by parameter adjustment both to satisfy Maxwell's equations (2) and to meet positive-definite probability-density requirements.

ION ORBIT EQUATIONS

For a rotary electromagnetic plane wave of vector potential amplitude a, propagating along a uniform axial magnetic field of magnitude b, the wave-frame vector potential $\underline{A}(\underline{R})$ has the form

$$\underline{A}(\underline{R}) = [a\cdot\cos(Kz)-b\cdot y/2,\ a\cdot\sin(Kz)+b\cdot x/2,\ 0,\ 0]. \qquad (5)$$

The complex Lorentz rotation relating the wave frame (unprimed coordinates) to the laboratory frame (primed coordinates) is given by

$$\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & K'/K & i\omega'/cK \\ 0 & 0 & -i\omega'/cK & K'/K \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix}, \quad \text{with} \quad K^2 \equiv (K')^2-(\omega'/c)^2, \qquad (6)$$

where K' and ω' are the laboratory-frame propagation vector and angular frequency respectively. Permitting the wave-frame propagation vector K to span the full complex field implies that the wave-frame (z,t) coordinates both may be complex, although the transverse (x,y) coordinates remain real.

Assembling the electromagnetic force-field tensor $[\hat{B}, \underline{E}/ic]$ by taking the tensor curl of the vector potential (5) yields

$$\nabla\times\underline{A}(\underline{R}) = \begin{pmatrix} 0 & b & Ka\ \sin Kz & 0 \\ -b & 0 & -Ka\ \cos Kz & 0 \\ -Ka\ \sin Kz & Ka\ \cos Kz & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (7)$$

The space-space components here are the magnetic force-field tensor $\hat{B}$. The space-time components are the null electric force-field vector $\underline{E}/ic$, whose vanishing here motivates the wave-frame construction. The ion orbit equations (3-4) thus become

$$dx/d\tau = p_x/m, \qquad dp_x/d\tau = (q/m)(p_y b + p_z Ka \sin Kz), \tag{8}$$

$$dy/d\tau = p_y/m, \qquad dp_y/d\tau = -\,(q/m)(p_x b + p_z Ka \cos Kz), \tag{9}$$

$$dz/d\tau = p_z/m, \qquad dp_z/d\tau = (q/m)Ka(-\,p_x \sin Kz + p_y \cos Kz), \tag{10}$$

$$dict/d\tau = iE/cm, \qquad diE/cd\tau = 0. \tag{11}$$

The ion species index s may be suppressed (as here), without loss of clarity, in any equation not involving summation over ion species. It is to be noted that the equations on the left indicate that real transverse (x,y) coordinates imply real transverse momenta (p_x, p_y) and that complex longitudinal (z,t) coordinates imply complex axial momentum p_z and complex energy E.

Constants of the Motion. The last equation states that the wave-frame ion energy E is a constant of the motion. Multiplying the $d(p_x, p_y, p_z)/d\tau$ equations by (p_x, p_y, p_z) respectively and adding them gives

$$p_x dp_x + p_y dp_y + p_z dp_z = 0, \tag{12}$$

so that wave-frame momentum magnitude $p = \surd(p_x^2 + p_y^2 + p_z^2)$ provides a second constant of the motion.

The structure of the transverse equations (8-9) suggests merging them in a transverse complex field, with $x + jy = re^{j\phi}$ and $p_x + jp_y = p_\perp e^{j\psi}$, to obtain as their transverse polar form,

$$d(re^{j\phi})/d\tau = p_\perp e^{j\psi}/m, \quad d(p_\perp e^{j\psi})/d\tau = -\,j(q/m)[bp_\perp e^{j\psi} + p_z Kae^{jKz}], \tag{13}$$

where $j = \surd(-1)$ has been used for the transverse complex field since $i = \surd(-1)$ lies in the longitudinal complex field. The last equation now integrates to

$$d[p_\perp e^{j\psi} + q(ae^{jKz} + jbre^{j\phi})] = 0, \tag{14}$$

which shows that the transverse components of the transverse quasi-momentum $p_\perp e^{j\psi} + q(ae^{jKz} + jbre^{j\phi})$ provide a third and fourth constant of the motion.

Inner Hamiltonian. The eight-dimensional orbit equations (8-11) possess seven constants of the motion, of which we now have four. A fifth constant of the motion, having the character of an inner Hamiltonian coupling the intrinsic physics of the problem, next is obtainable by resolving the transverse momentum equation (13) parallel and normal to $e^{j\psi}$ to give

$$dp_\perp/d\tau = (q/m)p_z Ka \sin(Kz - \psi), \tag{15}$$

$$d\psi/d\tau = -(q/m)[b + (p_z/p_\perp)Ka \cos(Kz - \psi)]. \tag{16}$$

Expressing $p_\perp$ in terms of p_z by use of $p_\perp^2 + p_z^2 = p^2$ converts the first transverse equation (15) to

$$dp_z/d\tau = -(q/m)p_\perp Ka \sin(Kz - \psi). \tag{17}$$

The coupling of the helical phase $(Kz - \psi)$ here suggests subtracting $d(Kz)/d\tau$ from both sides of equation (16) to obtain

$$d(Kz -\psi)/d\tau = (K/m)p_z + (q/m)[b + (p_z/p_\perp)Ka \cos(Kz -\psi)]. \tag{18}$$

The transverse momentum equations (17-18) consist now of two equations in two unknowns, the helical phase $(Kz - \psi)$ and the axial momentum p_z.

In this two-dimensional subspace, we can use the fact that any two-dimensional differential form possesses an integrating factor. Ratioing equations (17-18) gives the differential form

$$(q/m)p_\perp Ka \sin(Kz -\psi)\, d(Kz -\psi)$$
$$+ \{(K/m)p_z + (q/m)[b + (p_z/p_\perp)Ka \cos(Kz -\psi)]\}\, dp_z = 0. \tag{19}$$

Differentiating the first differential coefficient with respect to p_z, using $p_\perp^2 + p_z^2 = p^2$, yields

$$-(q/m)(p_z/p_\perp)Ka \sin(Kz -\psi),$$

and mentally differentiating the second differential coefficient with

respect to $(Kz - \psi)$ yields precisely this same result. The integrating factor thus is unity, and the differential form (19) integrates directly to provide our fifth constant of the motion,†

$$d[(p_z + qb/K)^2/2m - (qa/m)p_\perp \cos(Kz - \psi)] = 0, \tag{20}$$

which has been scaled to the dimensions of energy to facilitate immediate physical interpretation. The structure of this inner energy constant shows that when (K, z, p_z) are all pure real or pure imaginary, each ion stably rides the rotary traveling electromagnetic wave, energetically trapped in its momentum-dependent helical potential well.

Further analysis is simplified considerably by shifting the four-momentum representation to polar coordinates, as defined by

$$\begin{aligned} E/mc^2 &= \cosh\Lambda, & p/mc &= \sinh\Lambda, \\ p_z/mc &= \sinh\Lambda \cos\theta, & p_\perp/mc &= \sinh\Lambda \sin\theta, \\ p_x/mc &= \sinh\Lambda \sin\theta \cos\psi, & & \\ p_y/mc &= \sinh\Lambda \sin\theta \sin\psi. & & \end{aligned} \tag{21}$$

Here Λ is the Lorentz angle through which an ion has been rotated from rest, θ is the polar pitch angle of the ion momentum with respect to the axial magnetic field, and ψ is the azimuthal angle of the ion momentum about the axial magnetic field. In the wave frame, both Lorentz angle Λ and pitch angle θ may be complex, but azimuth angle ψ is real.

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† The existence of this constant of the motion has been noted previously by Laird[20].

In any two-dimensional subspace, it is well known that any constant of the motion can be used as a Hamiltonian by adroit choice of canonical conjugates. The polar momentum representation suggests dividing the inner energy (20) by p^2/m to define the inner Hamiltonian h as the dimensionless ratio,

$$\boxed{h \equiv (\cos\theta + qb/Kp)^2/2 - (qa/p)\sin\theta \cos(Kz - \psi),} \tag{22}$$

and then rescaling the transverse momentum equations (17-18) to dimensionless ratio form, to obtain

$$\begin{aligned} d(\cos\theta)/d(Kp\tau/m) &= -(qa/p)\sin\theta \sin(Kz - \psi), && (23)\\ &= -\partial h(Kz - \psi, \cos\theta)/\partial(Kz - \psi), \end{aligned}$$

$$\begin{aligned} d(Kz - \psi)/d(Kp\tau/m) &= (\cos\theta + qb/Kp) + (qa/p)\cot\theta \cos(Kz - \psi), && (24)\\ &= \partial h(Kz - \psi, \cos\theta)/\partial(\cos\theta). \end{aligned}$$

It becomes clear that $h(Kz - \psi, \cos\theta)$ can be used as an inner Hamiltonian function, with $(Kz - \psi, \cos\theta, Kp\tau/m)$ respectively taken as dimensionless (position, momentum, time) coordinates.

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This point in our analysis logic marks the outer boundary of previously existing firm theory, as delineated by our literature search for recent related work referenced above as well as others[28-36].

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FIRST INTEGRALS

Axial Momentum. The inner Hamiltonian equations of motion (23-24) consist of two equations involving two unknowns, the helical phase $(Kz - \psi)$ and the wave-frame pitch cosine $(\cos\theta)$. Maneuvering toward one equation in one unknown leads to eliminating the helical phase $(Kz - \psi)$ by combining the Hamiltonian constant (22) with the square of the pitch cosine derivative (23), to obtain

$$[d(\cos\theta)/d(Kp\tau/m)]^2 = (qa/p)^2(\sin^2\theta)[1 - \cos^2(Kz - \psi)],$$
$$= (qa/p)^2(1 - \cos^2\theta) - [(\cos\theta + qb/Kp)^2/2 - h]^2.$$

We now have one equation involving one unknown, the pitch angle cosine $(\cos\theta)$. Simplifying the most complicated term with an origin shift yields, sequentially,

$$u \equiv \cos\theta + qb/Kp, \tag{25}$$
$$[du/d(Kp\tau/m)]^2 = (qa/p)^2[1 - (u - qb/Kp)^2] - (u^2/2 - h)^2,$$
$$[du/d(Kp\tau/m)]^2 = - u^4/4 + [h - (qa/p)^2]u^2 + 2(qa/p)^2(qb/Kp)u$$
$$+ (qa/p)^2[1 - (qb/Kp)^2] - h^2. \tag{26}$$

The square of the derivative of u on the left here, with the fourth degree polynomial in u on the right, leads to immediate recognition that this equation is soluble in closed form in terms of doubly periodic Weierstrass functions.[37-41]

Maneuvering toward the Weierstrass differential equation,

$$[dP(z)/dz]^2 = 4(P-e_1)(P-e_2)(P-e_3), \quad e_1+e_2+e_3 = 0, \quad P(z\approx 0) \simeq 1/z^2, \tag{27}$$

by first factoring the quartic polynomial on the right of equation (26) and then factoring out one of the factors, yields the reduction

$$[du/d(iKp\tau/2m)]^2 = (u-u_1)(u-u_2)(u-u_3)(u-u_4), \tag{28}$$

$$[-d(u-u_4)^{-1}/d(iKp\tau/2m)]^2 = [(u-u_1)/(u-u_4)][(u-u_2)/(u-u_4)][(u-u_3)/(u-u_4)],$$

$$= [1-(u_1-u_4)/(u-u_4)][1-(u_2-u_4)/(u-u_4)][1-(u_3-u_4)/(u-u_4)].$$

The cubic polynomial structure here involving $(u-u_4)^{-1}$ suggests scaling to

$$w \equiv -(u_1-u_4)(u_2-u_4)(u_3-u_4)/4(u-u_4), \quad \text{to obtain} \tag{29}$$

$$[dw/d(iKp\tau/2m)]^2 = 4(w-w_1)(w-w_2)(w-w_3), \quad \text{where} \tag{30}$$

$$w_1 = -(u_2-u_4)(u_3-u_4)/4, \; w_2 = -(u_3-u_4)(u_1-u_4)/4, \; w_3 = -(u_1-u_4)(u_2-u_4)/4. \tag{31}$$

An origin shift to the centroid of the cubic roots,

$$\bar{w} = (w_1+w_2+w_3)/3, \quad e_1 = w_1-\bar{w}, \quad e_2 = w_2-\bar{w}, \quad e_3 = w_3-\bar{w}, \tag{32}$$

now yields the Weierstrass equation (27) with

$$P(iKp\tau/2m+\lambda) = w-\bar{w}, \tag{33}$$

where λ is a flexible integration constant. Backtracking through the preceding transformation hierarchy to assemble the first integral for the ion orbits yields, finally,

$$\cos\theta + qb/Kp = u_4 - (u_1-u_4)(u_2-u_4)(u_3-u_4)/4[P(iKp\tau/2m+\lambda) + \bar{w}]. \tag{34}$$

The quartic roots (u_1,u_2,u_3,u_4) obviously can be related to the coefficients of the quartic polynomial on the right of equation (26) by the symmetric polynomial forms

$$u_1+u_2+u_3+u_4 = 0, \tag{35}$$

$$u_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+u_3u_4 = 4[(qa/p)^2-h], \tag{36}$$

$$u_1u_2u_3+u_1u_2u_4+u_1u_3u_4+u_2u_3u_4 = 8(qa/p)^2(qb/Kp), \tag{37}$$

$$u_1u_2u_3u_4 = 4\{h^2+(qa/p)^2[(qb/Kp)^2-1]\}. \tag{38}$$

Parameter Inversion. The powerful general theory of doubly periodic functions can be used to clean up the curiously asymmetric and computationally awkward parameterization appearing in the solution (34). The structure of this doubly periodic solution displays two first order poles in each primitive period cell, situated at the two (and only two) primitive roots of

$$P(\delta) \equiv -\bar{w}, \quad \text{which lie at} \quad iKp\tau/2m + \lambda = \pm\delta. \tag{39}$$

Combining (31,32,39) with (27) yields the parametric interrelationship,

$$-[(u_1-u_4)(u_2-u_4)(u_3-u_4)]^2/4^3 = [e_1-P(\delta)][e_2-P(\delta)][e_3-P(\delta)],$$
$$= -[P'(\delta)]^2/4, \quad \text{or}$$

$$(u_1-u_4)(u_2-u_4)(u_3-u_4)/4 = -P'(\delta). \tag{40}$$

The choice of sign in the last step completes the definition of the bipolar separation (2δ) by sharpening the ($\pm$) choice left by (39). Our solution (34) thus can be parameterized more neatly now as

$$\cos\theta + qb/Kp = u_4 + P'(\delta)/[P(iKp\tau/2m+\lambda) - P(\delta)], \tag{41}$$

with δ parametrically replacing both $\bar{w}$ and $(u_1-u_4)(u_2-u_4)(u_3-u_4)$ via equations (39-40).

Shifting the remaining parameterization to the symmetric Weierstrass parameters (e_1,e_2,e_3), in an effort to get rid of the quartic root u_4 surviving as an obnoxious asymmetry in the solution (41), leads to combining

equations (31,32,39) to obtain

$$e_1-P(\delta) = -(u_2-u_4)(u_3-u_4)/4,$$

$$e_2-P(\delta) = -(u_3-u_4)(u_1-u_4)/4,$$

$$e_3-P(\delta) = -(u_1-u_4)(u_2-u_4)/4,$$

and then making use of equation (40) to extract

$$u_1-u_4 = P'(\delta)/[e_1-P(\delta)], \tag{42}$$

$$u_2-u_4 = P'(\delta)/[e_2-P(\delta)], \tag{43}$$

$$u_3-u_4 = P'(\delta)/[e_3-P(\delta)]. \tag{44}$$

Adding these three equations, using $u_1+u_2+u_3+u_4 = 0$, yields finally,

$$u_4 = \{P'(\delta)/[P(\delta)-e_1]+P'(\delta)/[P(\delta)-e_2]+P'(\delta)/[P(\delta)-e_3]\}/4,$$

$$= \{\ln [P'(\delta)]^2\}'/4, \quad \text{from equation (27);} \quad \text{hence}$$

$$u_4 = P''(\delta)/2P'(\delta), \tag{45}$$

and from (42-44) then,

$$u_1 = P''(\delta)/2P'(\delta) - P'(\delta)/[P(\delta)-e_1], \tag{46}$$

$$u_2 = P''(\delta)/2P'(\delta) - P'(\delta)/[P(\delta)-e_2], \tag{47}$$

$$u_3 = P''(\delta)/2P'(\delta) - P'(\delta)/[P(\delta)-e_3]. \tag{48}$$

Double differentiation of the Weierstrass half-period addition theorem[38(0.79)†], with implicit cyclic (1 2 3) index permutation, yields, sequentially,

$$[P(z+\omega_1)-e_1][P(z)-e_1] = (e_2-e_1)(e_3-e_1),$$

$$P'(z+\omega_1)/[P(z+\omega_1)-e_1]+P'(z)/[P(z)-e_1] = 0,$$

$$P''(z+\omega_1)/P'(z+\omega_1) = P''(z)/P'(z)-2P'(z)/[P(z)-e_1]. \tag{49}$$

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†Reference 38 (Equation 0.79).

Here $(\omega_1,\omega_2,\omega_3)$ are the Weierstrass half periods, with

$$P(\omega_1) = e_1, \quad P(\omega_2) = e_2, \quad P(\omega_3) = e_3, \quad \omega_1+\omega_2+\omega_3 = 0.$$

The quartic roots (45-48) thus reduce neatly to

$$u_1 = P''(\delta+\omega_1)/2P'(\delta+\omega_1), \tag{50}$$

$$u_2 = P''(\delta+\omega_2)/2P'(\delta+\omega_2), \tag{51}$$

$$u_3 = P''(\delta+\omega_3)/2P'(\delta+\omega_3), \tag{52}$$

$$u_4 = P''(\delta)/2P'(\delta), \tag{53}$$

and our bipolar solution (41) now becomes

$$\cos\theta + qb/Kp = P''(\delta)/2P'(\delta) + P'(\delta)/[P(iKp\tau/2m+\lambda) - P(\delta)]. \tag{54}$$

Further integration is facilitated by shifting this representation to the Weierstrass additively quasi-periodic zeta function, as defined by

$$d\zeta(z)/dz = -P(z), \quad \zeta(z\simeq 0) \simeq 1/z, \tag{55}$$

by use of the transparent identity[38(0.908$_2$)]

$$P'(z_0)/[P(z) - P(z_0)] = \zeta(z-z_0) - \zeta(z+z_0) + 2\zeta(z_0),$$

which yields

$$\cos\theta + qb/Kp = \zeta(iKp\tau/2m+\lambda-\delta) - \zeta(iKp\tau/2m+\lambda+\delta) + 2\zeta(\delta) + P''(\delta)/2P'(\delta).$$

Use of the further identity[33(18.4.7)]

$$P''(z)/2P'(z) = \zeta(2z) - 2\zeta(z),$$

reduces our bipolar doubly periodic solution to its most useful form, optimally parameterized for both theoretical investigation and numeric evaluation,

$$\boxed{\cos\theta + qb/Kp = \zeta(iKp\tau/2m+\lambda-\delta) - \zeta(iKp\tau/2m+\lambda+\delta) + \zeta(2\delta).} \tag{56}$$

This key first integral, in brief resumé, gives the wave-frame pitch-angle cosine ($\cos\theta$) at ion proper time τ as a bipolar doubly periodic function expressed explicitly as a phase-shift difference of the doubly quasi-periodic Weierstrass zeta function. The primitive (tiniest) periods ($2\omega_1, 2\omega_2$) and the primitive (tiniest) bipole separation (2δ) are parametrically related to the three coefficients ($h, qa/p, qb/Kp$) appearing in the differential equation of motion (26) by the cascade of boxed equations (35-38,50-53).

<u>Transverse Momentum</u>. Transverse momentum-projection first integrals now can be generated from the inner Hamiltonian (22) and the transverse equation of motion (23), respectively, as the rotary orthogonal projection cosines,

$$\sin\theta \cos(Kz - \psi) = (p/qa)[(\cos\theta + qb/Kp)^2/2 - h], \qquad (57)$$

$$\sin\theta \sin(Kz - \psi) = - (p/qa)\, d(\cos\theta)/d(Kp\tau/m). \qquad (58)$$

Forming the square of the axial first integral (56) for subsequent insertion in (57), by Laurent expansion about either pole, yields as leading terms,

$$\cos\theta + qb/Kp \simeq \pm 1/(iKp\tau/2m+\lambda\mp\delta) \pm P(2\delta)\cdot(iKp\tau/2m+\lambda\mp\delta),$$

$$(\cos\theta + qb/Kp)^2 \simeq 1/(iKp\tau/2m+\lambda\mp\delta)^2 + 2P(2\delta).$$

From the general theory of doubly periodic functions, it follows that

$$(\cos\theta + qb/Kp)^2 = P(iKp\tau/2m+\lambda-\delta) + P(iKp\tau/2m+\lambda+\delta) + P(2\delta). \qquad (59)$$

Differentiating the axial first integral (56) for subsequent insertion in (58) yields at once,

$$d(\cos\theta)/d(Kp\tau/m) = - (i/2)[P(iKp\tau/2m+\lambda-\delta) - P(iKp\tau/2m+\lambda+\delta)]. \qquad (60)$$

Assembling these pieces now gives the transverse momentum-projection first integrals (57-58) as the bipolar doubly periodic functions of ion proper time τ,

$$\boxed{\begin{aligned} \sin\theta\,\cos(Kz-\psi) &= (p/2qa)[P(iKp\tau/2m+\lambda-\delta)+P(iKp\tau/2m+\lambda+\delta)+P(2\delta)-2h], \\ \sin\theta\,\sin(Kz-\psi) &= i(p/2qa)[P(iKp\tau/2m+\lambda-\delta)-P(iKp\tau/2m+\lambda+\delta)]. \end{aligned}} \begin{matrix} (61) \\ (62) \end{matrix}$$

The astonishing structural similarity of these rotary orthogonal projectors suggests multiplying the second by i and then adding and subtracting the two to obtain the cisoidal resolution,

$$\boxed{\begin{aligned} \sin\theta\, e^{i(Kz-\psi)} &= (p/qa)[P(iKp\tau/2m+\lambda+\delta)+P(2\delta)/2-h], \\ \sin\theta\, e^{-i(Kz-\psi)} &= (p/qa)[P(iKp\tau/2m+\lambda-\delta)+P(2\delta)/2-h]. \end{aligned}} \qquad \begin{matrix} (63) \\ (64) \end{matrix}$$

Ratioing this cisoidal pair extracts the helical-phase first integral as

$$\boxed{e^{i2(Kz-\psi)} = [P(iKp\tau/2m+\lambda+\delta)+P(2\delta)/2-h]/[P(iKp\tau/2m+\lambda-\delta)+P(2\delta)/2-h],} \quad (65)$$

to display the explicit solution for the second inner Hamiltonian equation of motion (24).

SECOND INTEGRALS

Axial Position. The first of the axial equations of motion (10) combines with the axial first integral (56) to give

$$d(Kz)/d(Kp\tau/m)+qb/Kp = \zeta(iKp\tau/2m+\lambda-\delta)-\zeta(iKp\tau/2m+\lambda+\delta)+\zeta(2\delta).$$

Integrating by use of the multiplicatively quasi-periodic Weierstrass sigma function, as defined by

$$[d\sigma(z)]/\sigma(z)dz = \zeta(z), \quad \sigma(z\approx 0) \simeq z, \text{ yields} \tag{66}$$

$$\boxed{e^{iK(z-z_0)} = e^{[\zeta(2\delta)-qb/Kp](iKp\tau/m)}\cdot\sigma^2(iKp\tau/2m+\lambda-\delta)\sigma^2(\lambda+\delta) /\sigma^2(iKp\tau/2m+\lambda+\delta)\sigma^2(\lambda-\delta).} \tag{67}$$

This axial second integral gives the wave-frame ion axial position $z(\tau)$ as an explicit doubly quasi-periodic function of ion proper time τ, with $z_0 \equiv z(0)$.

From the multiplicative double quasi-periodicity of the Weierstrass sigma function[38(0.73)]

$$\sigma(z+\nu_1\omega_1+\nu_2\omega_2) = - e^{2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]z}\,\sigma(z-\nu_1\omega_1-\nu_2\omega_2), \tag{68}$$

with (ν_1,ν_2) spanning all positive and negative integers except the (0,0) pair, it follows that our axial position integral (67) has the multiplicative double quasi-periodicity

$$e^{iK[z(iKp\tau/2m+\nu_1\omega_1+\nu_2\omega_2)-z(iKp\tau/2m-\nu_1\omega_1-\nu_2\omega_2)]} = e^{4[\zeta(2\delta)-qb/Kp](\nu_1\omega_1+\nu_2\omega_2)-8[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\delta}. \tag{69}$$

The helical concept of axial pitch thus can be ascribed to this discrete doubly-indexed subset of ion orbits, as given explicitly by

$$z(iKp\tau/2m+\nu_1\omega_1+\nu_2\omega_2)-z(iKp\tau/2m-\nu_1\omega_1-\nu_2\omega_2) = \{4[\zeta(2\delta)-qb/Kp](\nu_1\omega_1+\nu_2\omega_2)-8[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\delta\}/iK. \tag{70}$$

This remarkable equation states that, as the ion momentum cycles through any doubly-indexed Weierstrass period $2\nu_1\omega_1+2\nu_2\omega_2$, the wave-frame ion axial position z changes by the fixed axial pitch appearing here on the right. A fascinating observation of fundamental relevance is that all ion orbits accessible to numeric evaluation on a digital computer are contained in this discrete doubly-indexed subset, since digital computer numbers span only discrete integer ratios.

Transverse Position. The transverse second integrals now can be assembled quite simply, by first pulling the transverse quasi-momentum constants of the motion (14) out of the transverse complex field in the (x,y) component form as

$$p \sin\theta \cos\psi + qa \cos Kz + qbr \cos(\phi+\pi/2),$$

$$p \sin\theta \sin\psi + qa \sin Kz + qbr \sin(\phi+\pi/2),$$

and then recombining them in cisoidal form in the longitudinal complex field as

$$p \sin\theta\, e^{i\psi} + qa\, e^{iKz} + qbr\, e^{i(\phi+\pi/2)},$$

$$p \sin\theta\, e^{-i\psi} + qa\, e^{-iKz} + qbr\, e^{-i(\phi+\pi/2)}.$$

Keeping in mind here that (p,θ,K,z) may be complex, and making use of the axial first and second integrals (63,64,67), yields for the transverse position integrals,

$$re^{i\phi} = r_0e^{i\phi_0} + i(a/b)(e^{iKz}-e^{iKz_0})$$
$$+ i(p/qb)\{[\sin\theta\, e^{-i(Kz-\psi)}]e^{iKz}-(p/qa)[P(\lambda-\delta)+P(2\delta)/2-h]e^{iKz_0}\}, \quad (71)$$

$$re^{-i\phi} = r_0e^{-i\phi_0} - i(a/b)(e^{-iKz}-e^{-iKz_0})$$
$$- i(p/qb)\{[\sin\theta\, e^{i(Kz-\psi)}]e^{-iKz}-(p/qa)[P(\lambda+\delta)+P(2\delta)/2-h]e^{-iKz_0}\}, \quad (72)$$

where the integration constants r_0,ϕ_0,z_0 refer to proper time $\tau = 0$ for each ion. The factors on the right [$\sin\theta\ e^{\pm i(Kz-\psi)}$ and $e^{\pm iKz}$] refer to the explicit functions of ion proper time τ given by the axial first and second integrals (63,64,67). The periodicity of the former and the quasi-periodicity of the latter suggest gathering all the constant terms to the left side of equations (71-72), as a complex gyration center, and shifting to complex polar coordinates (R,Φ) about this gyration center, as defined by

$$Re^{i\Phi} = re^{i\phi}-r_0e^{i\phi_0}+i(a/b)\{1+(p/qa)^2[P(\lambda-\delta)+P(2\delta)/2-h]\}e^{iKz_0}, \qquad (73)$$

$$Re^{-i\Phi} = re^{-i\phi}-r_0e^{-i\phi_0}-i(a/b)\{1+(p/qa)^2[P(\lambda+\delta)+P(2\delta)/2-h]\}e^{-iKz_0}, \qquad (74)$$

to reveal the functional structure

$$Re^{i\Phi} = i(a/b)[1 + (p/qa)\sin\theta\ e^{-i(Kz-\psi)}]e^{iKz},$$

$$Re^{-i\Phi} = -i(a/b)[1 + (p/qa)\sin\theta\ e^{i(Kz-\psi)}]e^{-iKz}.$$

Explicit insertion of the axial first integrals (63,64) for $\sin\theta\ e^{\pm i(Kz-\psi)}$ converts these transverse second integrals to the more readily interpretable quasi-helical form,

$$\boxed{\begin{aligned} Re^{i(\Phi-Kz)} &= i(a/b)\{1+(p/qa)^2[P(iKp\tau/2m+\lambda-\delta)+P(2\delta)/2-h]\}, \\ Re^{-i(\Phi-Kz)} &= -i(a/b)\{1+(p/qa)^2[P(iKp\tau/2m+\lambda+\delta)+P(2\delta)/2-h]\}. \end{aligned}} \qquad \begin{aligned}(75)\\(76)\end{aligned}$$

The helical geometric structure (R,Φ,z) on the left combined with the bipolar doubly periodic proper time (τ) structure on the right show that the ion orbits do have the primary structure of helices but also have multi-lobe oscillatory secondary spiral structure superimposed on the helix. The actual physical shape of these double helix orbits depends, of course, on the relative magnitudes of the constants.

The striking similarity of these helical spiral ion orbits to the spiral staircase quantum structure of biophysics suggests that the presence of a planetary magnetic field may play an essential role in the evolution and reproduction of life, by providing a magnetic axis about which ions tend to cluster in a spiral staircase.

Forming the product and ratio of the transverse second integrals (75,76) extracts both the transverse gyration radius R and its helical azimuth Φ-Kz as the explicit multi-polar doubly periodic functions of ion proper time τ,

$$R^2 = (a/b)^2\{1+(p/qa)^2[P(iKp\tau/2m+\lambda-\delta)+P(2\delta)/2-h]\}$$
$$\cdot\{1+(p/qa)^2[P(iKp\tau/2m+\lambda+\delta)+P(2\delta)/2-h]\}, \qquad (77)$$
$$e^{i2(\Phi-Kz)} = -\{1+(p/qa)^2[P(iKp\tau/2m+\lambda-\delta)+P(2\delta)/2-h]\}$$
$$/\{1+(p/qa)^2[P(iKp\tau/2m+\lambda+\delta)+P(2\delta)/2-h]\}. \qquad (78)$$

ION DISTRIBUTION

Viewing the eight-dimensional Vlasov differential operator appearing in equation (1) as simply the ion proper time derivative,

$$dn_s(\underline{R}(\tau),\underline{P}(\tau))/d\tau = 0, \tag{79}$$

makes it clear that its complete solution consists of any function of any seven functionally independent constants of the ion-orbit motion. Five constants of the ion motion have been displayed explicitly in (11,12,14,22) as $[E,p,p_\perp e^{j\psi}+q(ae^{jKz}+jbre^{j\phi}),h]$, and two more have emerged implicitly in (56,67) as the first and second axial integration constants (λ,z_0). Assembling all seven constants of the ion motion here for subsequent reference, as explicit or implicit functions of the ion phase-space coordinates $(x,y,z,ct;mc,\Lambda,\theta,\psi)$, yields in our complex wave frame:

Constants of Ion Motion (Wave Frame)

Ion Energy (E): From (11,21), $E \equiv mc^2 \cosh\Lambda.$ (80)

Ion Momentum Magnitude (p): From (12,21), $p \equiv mc \sinh\Lambda.$ (81)

Ion Transverse Gyro Center (X_θ,Y_θ): From (14,21),

$$X_\theta \equiv x + (mc/qb) \sinh\Lambda \sin\theta \sin\psi + (a/b) \sin Kz, \tag{82}$$

$$Y_\theta \equiv y - (mc/qb) \sinh\Lambda \sin\theta \cos\psi - (a/b) \cos Kz. \tag{83}$$

Ion Inner Hamiltonian (h): From (21,22),

$$h \equiv [\cos\theta + qb/(Kmc \sinh\Lambda)]^2/2 - [qa/(mc \sinh\Lambda)] \sin\theta \cos(Kz-\psi). \tag{84}$$

First and Second Axial Integration Constants (λ,z_0):

From (21,56,67), using left (11) integral $Kp\tau/m = Kct \tanh\Lambda$,

$$\cos\theta + qb/(Kmc \sinh\Lambda) \equiv \zeta((iKct/2)\tanh\Lambda+\lambda-\delta) - \zeta((iKct/2)\tanh\Lambda+\lambda+\delta) + \zeta(2\delta), \tag{85}$$

$$e^{iKz_0} \equiv e^{iKz-[\zeta(2\delta)-qb/(Kmc \sinh\Lambda)]iKct \tanh\Lambda} \cdot\sigma^2((iKct/2)\tanh\Lambda+\lambda+\delta)\cdot\sigma^2(\lambda-\delta)/\sigma^2((iKct/2)\tanh\Lambda+\lambda-\delta)\cdot\sigma^2(\lambda+\delta). \tag{86}$$

The first axial integral (85) gives the first axial integration constant as an implicit function $\lambda = \lambda(ct,mc,\Lambda,\theta)$. The second axial integral (86) then gives the second axial integration constant as a mixed explicit-implicit function $z_0 = z_0(z,ct,mc,\Lambda,\lambda(ct,mc,\Lambda,\theta))$. Forming the accessible momentum phase element from (21,80,81) as

$$dp_x dp_y dp_z dE/c = \delta(m'c-mc)\ (m'c)^3 d(m'c)\ \sinh^2\Lambda d\Lambda\ \sin\theta d\theta\ d\psi, \tag{87}$$

now yields as the complete probability density element for the Vlasov ion plasma,

$$\boxed{d^6N/dxdydz = n(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)\ \sinh^2\Lambda d\Lambda\ \sin\theta d\theta\ d\psi,} \tag{88}$$

where the flexible ion phase-space distribution $n(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)$ spans the six constants of the motion remaining after meeting the delta-function rest-mass constraint in (87).

Development Sequence. The establishment of a general distribution in incorporating all of these available constants is a formidable task. Certain dependences are ignored initially. For example, it is important to recognize that the dependence of the ion distribution upon the gyro-center (X_θ,Y_θ) constants (82-83) implies transverse (x,y) variation of the ion density, and that dependence of the ion distribution upon the implicit first or second axial integration constants (λ,z_0) of (85-86) implies intractably intricate longitudinal (ct,[z,ct]) variation of the wave-frame ion density. The complementary inference here, then, is that the transversely uniform subgroup of rotational modes which stably ride the rotary electromagnetic wave with no wave-frame time variation should be given by an ion distribution dependent upon only Lorentz angle Λ and inner Hamiltonian h, of the sort

$$d^6N/dxdydz = n(\Lambda,h)\ \sinh^2\Lambda d\Lambda\ \sin\theta d\theta\ d\psi. \tag{89}$$

Realistic further theoretical investigation should perhaps focus first upon this latter form, as a fundamental mode apt to be of dominant importance in most physical applications.

Subsequent generalization might logically focus next upon transversely variant modes dependent also on the gyro-center (X_θ, Y_θ) constants, of the sort

$$\boxed{d^6N_s/dxdydz = n_s(X_\theta, Y_\theta, \Lambda, h)\ \sinh^2\Lambda d\Lambda\ \sin\theta d\theta\ d\psi.} \tag{90}$$

Transversely variant modes like this quite certainly are needed to investigate fusion containment leakage through the lateral wall of a magnetic bottle.

As a later development phase, substantial improvements in ion-plasma perturbation theory should be obtainable in tractable form from the complete distribution (88), with direct inclusion of the intricately coupled stability effects of perturbing all the constants of the ion motion $(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda)$.

Finally, nothing more formidable than cascaded implicit transformation algebra seems to be needed to fit the complete distribution (88) directly to an arbitrary laboratory-frame initial distribution, with subsequent generation of the full time-dependent evolution of the laboratory-frame ion distribution in closed form.

ELECTROMAGNETIC FIELD EQUATIONS

Ion-Orbit Current and Charge Integrals. For the axially-guided rotary-wave vector potential (5) and the corresponding complete Vlasov ion distribution (88), Maxwell's equations (2) become

$$K^2 a \cos Kz = \mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \sinh^3\Lambda d\Lambda \sin^2\theta d\theta \cos\psi d\psi, \tag{91}$$

$$K^2 a \sin Kz = \mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \sinh^3\Lambda d\Lambda \sin^2\theta d\theta \sin\psi d\psi, \tag{92}$$

$$0 = \mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \sinh^3\Lambda d\Lambda \cos\theta \sin\theta d\theta\, d\psi, \tag{93}$$

$$0 = \mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \cosh\Lambda \sinh^2\Lambda d\Lambda \sin\theta d\theta\, d\psi. \tag{94}$$

Here the ion distribution function $n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda)$ is complete, the functions (X_θ, Y_θ, h) are given explicitly by (82-84), the functions (λ, z_0) are given implicitly by (85-86), the complex integration domain for (Λ, θ) is obtainable by backtracking to the laboratory frame via (6), the azimuthal (ψ) integration spans $(-\pi<\psi<\pi)$, and the summation on s spans all ion species present. Resolving the transverse (x,y) equations (91-92) parallel and normal to the rotary vector potential, by multiplying the pair from the left by the rotation matrix

$$\begin{pmatrix} \cos Kz & \sin Kz \\ -\sin Kz & \cos Kz \end{pmatrix},$$ gives the simpler transverse pair,

$$K^2 a = \mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \sinh^3\Lambda d\Lambda \sin^2\theta d\theta \cos(Kz-\psi) d\psi, \tag{95}$$

$$0 = -\mu c \sum_s q_s \int n_s(X_\theta, Y_\theta, z_0, \Lambda, h, \lambda) \sinh^3\Lambda d\Lambda \sin^2\theta d\theta \sin(Kz-\psi) d\psi. \tag{96}$$

All four equations (93-96) now can be integrated explicitly by shifting to ion orbit coordinates with the differential transformation,

$$\sin\theta d\theta\, d\psi = d(Kz-\psi)\, d(\cos\theta) = |\partial(Kz-\psi,\cos\theta)/\partial(Kp\tau/m,h)|\, d(Kp\tau/m)\, dh.$$

The transformation determinant here is unity, for the inner Hamiltonian equations (23,24) evaluate it quite directly as

$$[\partial(Kz-\psi)/\partial(Kp\tau/m)][\partial(\cos\theta)/\partial h]-[\partial(\cos\theta)/\partial(Kp\tau/m)][\partial(Kz-\psi)/\partial h]$$

$$= [\partial h/\partial(\cos\theta)][\partial(\cos\theta)/\partial h]+[\partial h/\partial(Kz-\psi)][\partial(Kz-\psi)/\partial h] = \partial h/\partial h = 1.$$

The differential transform to ion orbit coordinates thus becomes simply

$$\sin\theta d\theta\, d\psi = d(Kz-\psi)\, d(\cos\theta) = d(Kp\tau/m)\, dh, \tag{97}$$

and making use of the transverse and axial first integrals (61-62,56) as they arise now converts Maxwell's equations (95-96,93-94) to their elegant ion-orbit integral form,

$$K^2a = (\mu c^2/2a)\textstyle\sum_s m_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)[P(iKp\tau/2m+\lambda-\delta)$$
$$+P(iKp\tau/2m+\lambda+\delta)+P(2\delta)-2h]\, d(Kp\tau/m)\, dh\, \sinh^4\Lambda d\Lambda, \tag{98}$$

$$0 = -i(\mu c^2/2a)\textstyle\sum_s m_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)[P(iKp\tau/2m+\lambda-\delta)$$
$$-P(iKp\tau/2m+\lambda+\delta)]\, d(Kp\tau/m)\, dh\, \sinh^4\Lambda d\Lambda, \tag{99}$$

$$0 = \mu c\textstyle\sum_s q_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)[\zeta(2\delta)-q_s b/(Km_s c\, \sinh\Lambda)$$
$$+\zeta(iKp\tau/2m+\lambda-\delta)-\zeta(iKp\tau/2m+\lambda+\delta)]\, d(Kp\tau/m)\, dh\, \sinh^3\Lambda d\Lambda, \tag{100}$$

$$0 = \mu c\textstyle\sum_s q_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)\, d(Kp\tau/m)\, dh\, \cosh\Lambda\, \sinh^2\Lambda d\Lambda. \tag{101}$$

From the doubly periodic structure of both the axial and transverse first integrals (56,61-62), it follows that the ion proper time integrals here must span one doubly-indexed Weierstrass period $2\nu_1\omega_1+2\nu_2\omega_2$, to properly span the momentum angular phase space once and only once.

The transverse current integral (99) normal to the rotary vector

potential manifestly vanishes, by integrating with the help of the zeta function (55) to a doubly periodic zeta function difference having the same value at both ends of the Weierstrass period. With the help of the zeta function (55) and its additive double quasi-periodicity[32(0.63₂)],

$$\zeta(z+\nu_1\omega_1+\nu_2\omega_2) = \zeta(z-\nu_1\omega_1-\nu_2\omega_2) + 2\nu_1\zeta(\omega_1) + 2\nu_2\zeta(\omega_2), \tag{102}$$

and with the help of the sigma function (66) and its multiplicative double quasi-periodicity (68), Maxwell's remaining three equations (98,100,101) integrate mentally now to the merged Vlasov-Maxwell ion-plasma triad,

$$\boxed{\begin{aligned} iK^2a^2/2\mu &= \textstyle\sum_s m_s c^2 \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)\{[P(2\delta)-2h](\nu_1\omega_1+\nu_2\omega_2) \\ &\quad -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\}\ dh\ \sinh^4\Lambda d\Lambda, & (103) \\ 0 &= \textstyle\sum_s q_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)\{[\zeta(2\delta)-q_s b/(Km_s c\ \sinh\Lambda)](\nu_1\omega_1+\nu_2\omega_2) \\ &\quad -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\delta\}\ dh\ \sinh^3\Lambda d\Lambda, & (104) \\ 0 &= \textstyle\sum_s q_s \int n_s(X_\theta,Y_\theta,z_0,\Lambda,h,\lambda)[\nu_1\omega_1+\nu_2\omega_2]\ dh\ \cosh\Lambda\ \sinh^2\Lambda d\Lambda. & (105) \end{aligned}}$$

All quantities appearing in these equations, except (K,a,b,μ,c), implicitly contain a suppressed ion-species label s. The first of this triad (103-105) relates the rotary electromagnetic wave amplitude to the rotary ion current source. The last two state, respectively, that both the axial ion current and the charge density are zero.

The ultra-intricate dependence of the Weierstrass primitive periods $(2\omega_1,2\omega_2)$ and of the primitive bipole separation (2δ) upon the three coefficients $[h,qa/(mc\ \sinh\Lambda),qb/(Kmc\ \sinh\Lambda)]$ in the axial equation of motion (26), via the cascade of boxed equations (35-38,50-53), makes further integration at this full level of completeness quite formidable.

<u>Generator Mode</u> (<u>BNW Program Plasma-III</u>). Critical scrutiny of the residual parametric integration structure of the merged Vlasov-Maxwell orbit-integrated triad (103-105) reveals the existence of one uniquely simple modal subgroup, which both admits full closed-form integration of equations (103-105) and yet contains just sufficient parametric flexibility to conceivably be developable as a generator function for all modes. The analysis logic for this generator mode (now operable on the Cyber-74 computer as Battelle-Northwest Program Plasma-III) begins with its double delta function parametric ion distribution,

$$\boxed{n_s(X_\theta, Y_\theta, z_0, \Lambda', h', \lambda) = n_s(X_\theta, Y_\theta, z_0, \lambda)\ \delta(\Lambda' - \Lambda)\ \delta(h' - h).} \tag{106}$$

For this ion distribution, the merged Vlasov-Maxwell orbit integrals (103-105) reduce to three algebraic equations,

$$iK^2a^2/2\mu = \sum_s m_s c^2 n_s(X_\theta, Y_\theta, z_0, \lambda)\{[P(2\delta) - 2h](\nu_1\omega_1 + \nu_2\omega_2) \\ -2[\nu_1\zeta(\omega_1) + \nu_2\zeta(\omega_2)]\}\sinh^4\Lambda, \tag{107}$$

$$0 = \sum_s q_s n_s(X_\theta, Y_\theta, z_0, \lambda)\{[\zeta(2\delta) - q_s b/(K m_s c \sinh\Lambda)](\nu_1\omega_1 + \nu_2\omega_2) \\ -2[\nu_1\zeta(\omega_1) + \nu_2\zeta(\omega_2)]\delta\}\sinh^3\Lambda, \tag{108}$$

$$0 = \sum_s q_s n_s(X_\theta, Y_\theta, z_0, \lambda)[\nu_1\omega_1 + \nu_2\omega_2]\cosh\Lambda\ \sinh^2\Lambda. \tag{109}$$

The first of this algebraic triad relates the rotary electromagnetic wave amplitude to the rotary ion current source. The last two state, respectively, that both the axial ion current and the charge density are zero. For a mixture of I ion species, parametric flexibility here spans the real pair (a,b) and the 4I+1 complex parameters $[K, (\underline{\omega}_1, \underline{\omega}_2, \underline{\delta}, \underline{\Lambda}) \equiv (\omega_1, \omega_2, \delta, \Lambda)_s]$, intricately interrelated by 2I-1 complex

constraints visible in (35-38,50-53) as the I ion-independent ratios,

$$\boxed{[q_s a/(m_s c\ \sinh\Lambda_s)]/[q_s b/(Km_s c\ \sinh\Lambda_s)] = Ka/b,} \tag{110}$$

and the I-1 ion-dependent ratios,

$$\boxed{[q_1 a/(m_1 c\ \sinh\Lambda_1)]/[q_s a/(m_s c\ \sinh\Lambda_s)] = (q_1 m_s\ \sinh\Lambda_s)/(q_s m_1\ \sinh\Lambda_1),} \tag{111}$$

to leave the net equivalent of 2I+3 complex parametric degrees of freedom for each 2I-tuply indexed double period mode $(\underline{\nu}_1,\underline{\nu}_2) \equiv (\nu_1,\nu_2)_s$. Exploring this vast subset of closed-form transversely-uniform solutions may take quite a while.

Three Ion Species. Brief scrutiny of the three algebraic equations (107-109) leads quickly to focusing initial investigation upon a mixture of three ion species [such as (D^+,T^+,e^-) for a deuterium-tritium fusion plasma, or (H^+,D^+,e^-) for a planetary or stellar proton plasma], and simply inverting the 3x3 coefficient matrix to obtain the parametric ion distribution set $\underline{n} \equiv (n_1,n_2,n_3)$ as

$$\boxed{\underline{n}(\underline{\Lambda},\underline{h}) = \begin{pmatrix} [mc^2\{[P(2\delta)-2h](\nu_1\omega_1+\nu_2\omega_2) \\ \quad -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\}\sinh^4\Lambda]_s \\ [q\{[\zeta(2\delta)-qb/(Kmc\ \sinh\Lambda)](\nu_1\omega_1+\nu_2\omega_2) \\ \quad -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\delta\}\sinh^3\Lambda]_s \\ [q(\nu_1\omega_1+\nu_2\omega_2)\cosh\Lambda\ \sinh^2\Lambda]_s \end{pmatrix}^{-1} \cdot \begin{pmatrix} iK^2a^2/2\mu \\ 0 \\ 0 \end{pmatrix},} \tag{112}$$

where the ion-species index s = (1,2,3) labels three such column

vectors comprising the matrix to be inverted. The arguments $(\underline{\Lambda},\underline{h})$ on the left stress the explicit appearance of the ion set $(\underline{\Lambda},\underline{h})$ on the right, as distinct from implicit dependence upon the parameter basis $(a,b,K,\underline{\omega_1},\underline{\omega_2},\underline{\delta})$.

Transformation of this ion distribution from the parametric basis back to the wave-frame (z,Λ,θ,ψ) basis involves now simply direct substitution of the ion inner Hamiltonian functions $h(z,\Lambda,\theta,\psi)$ of (84) for the inner Hamiltonian parameter set $\underline{h}$ on the right side of the last equation. No density transformation determinant accompanies such a shift from the parametric basis to the wave-frame basis, since the complete parametric solution (88) was defined in the wave frame. For each hextuply indexed double period mode $(\underline{\nu_1},\underline{\nu_2})$, these transversely uniform triple ion solutions span nine complex parametric degrees of freedom, according to the logic in the vicinity of (110,111).

Four or more ion species can be handled quite similarly, by inverting any 3x3 coefficient sub-matrix appearing in the three algebraic equations (107-109) to express any three ion distributions in terms of the other ion distributions.

<u>Two Ion Species</u>. Two ion species emerge from the three algebraic equations (107-109) as a curious determinantal degeneracy, which can be read off the last two homogeneous equations (108-109) by setting the determinant of their coefficient matrix to zero to obtain

$$\boxed{\begin{vmatrix} [\{[\zeta(2\delta)-qb/(Kmc\ \sinh\Lambda)](\nu_1\omega_1+\nu_2\omega_2) \\ -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\delta\}\sinh\Lambda]_s \\ \\ [(\nu_1\omega_1+\nu_2\omega_2)\cosh\Lambda]_s \end{vmatrix} = 0,} \qquad (113)$$

where the ion species index $s = (1,2)$ labels two such column vectors comprising the 2x2 singular matrix. The parametric ion distribution next is obtainable by inverting the 2x2 coefficient matrix for the inhomogeneous equation (107) combined with either of the now coalescent homogeneous equations (108-109), to yield the set $\underline{n} \equiv (n_1, n_2)$ as

$$\boxed{\underline{n}(\underline{\Lambda},\underline{h}) = \begin{pmatrix} [mc^2\{[P(2\delta)-2h](\nu_1\omega_1+\nu_2\omega_2) \\ \qquad -2[\nu_1\zeta(\omega_1)+\nu_2\zeta(\omega_2)]\}\sinh^4\Lambda]_s \\ \\ [q(\nu_1\omega_1+\nu_2\omega_2)\cosh\Lambda\ \sinh^2\Lambda]_s \end{pmatrix}^{-1} \cdot \begin{pmatrix} iK^2a^2/2\mu \\ \\ \\ 0 \end{pmatrix},} \qquad (114)$$

where again the ion species index $s = (1,2)$ labels two such column vectors comprising the 2x2 matrix to be inverted. For each quadruply indexed double period mode $(\underline{\nu}_1, \underline{\nu}_2)$, these transversely uniform double ion solutions span six complex parametric degrees of freedom, according to the logic in the vicinity of (110,111) as modified to include the degeneracy constraint (113).

NUMERICS

Fully optimized digital computer generation of all the preceding doubly periodic functions and their first and second integrals proceeds [via Whittaker and Watson[37(Ch. XXI)]] by first summing to solid (60-bit) convergence thirteen complex Fourier series defining the four doubly quasi-periodic Jacobi theta functions,

$$\theta_1(z) \equiv 2\textstyle\sum_{n=0}^{\infty} (-)^n \exp[i\pi\hat{\tau}(n+1/2)^2] \sin[(2n+1)z], \tag{115}$$
$$\theta_2(z) \equiv 2\textstyle\sum_{n=0}^{\infty} \exp[i\pi\hat{\tau}(n+1/2)^2] \cos[(2n+1)z], \tag{116}$$
$$\theta_3(z) \equiv 1 + 2\textstyle\sum_{n=1}^{\infty} \exp[i\pi\hat{\tau}n^2] \cos[2nz], \tag{117}$$
$$\theta_4(z) \equiv 1 + 2\textstyle\sum_{n=1}^{\infty} (-)^n \exp[i\pi\hat{\tau}n^2] \cos[2nz], \tag{118}$$

their four first derivatives,

$$\theta_1'(z) = 4\textstyle\sum_{n=0}^{\infty} (-)^n \exp[i\pi\hat{\tau}(n+1/2)^2]\,(n+1/2) \cos[(2n+1)z], \tag{119}$$
$$\theta_2'(z) = -4\textstyle\sum_{n=0}^{\infty} \exp[i\pi\hat{\tau}(n+1/2)^2]\,(n+1/2) \sin[(2n+1)z], \tag{120}$$
$$\theta_3'(z) = -4\textstyle\sum_{n=1}^{\infty} \exp[i\pi\hat{\tau}n^2]\, n \sin[2nz], \tag{121}$$
$$\theta_4'(z) = -4\textstyle\sum_{n=1}^{\infty} (-)^n \exp[i\pi\hat{\tau}n^2]\, n \sin[2nz], \tag{122}$$

and their nodal normalizer quintet,

$$\theta_1'(0) = 4\textstyle\sum_{n=0}^{\infty} (-)^n \exp[i\pi\hat{\tau}(n+1/2)^2]\,(n+1/2), \tag{123}$$
$$\theta_1'''(0) = -16\textstyle\sum_{n=0}^{\infty} (-)^n \exp[i\pi\hat{\tau}(n+1/2)^2]\,(n+1/2)^3, \tag{124}$$
$$\theta_2(0) = 2\textstyle\sum_{n=0}^{\infty} \exp[i\pi\hat{\tau}(n+1/2)^2], \tag{125}$$
$$\theta_3(0) = 1 + 2\textstyle\sum_{n=1}^{\infty} \exp[i\pi\hat{\tau}n^2], \tag{126}$$
$$\theta_4(0) = 1 + 2\textstyle\sum_{n=1}^{\infty} (-)^n \exp[i\pi\hat{\tau}n^2]. \tag{127}$$

Here the primitive (tiniest) complex periods $(2\omega_1, 2\omega_2)$ are to be ordered

in phase angle (by relabeling, or by replacing either period with its negative) so that the primitive period ratio $\hat{\tau} \equiv \omega_2/\omega_1$ has a positive imaginary part to provide uniform series convergence. Because of the presence of the double-exponential attenuation factor $\exp[i\pi\hat{\tau}n^2]$, all these series display ultra-rapid convergence over the full complex field, except quite near their dense line of intrinsic essential singularities along $\mathrm{Im}(\hat{\tau}) \approx 0$.

The doubly periodic Weierstrass pay function $P(z)$, the additive doubly quasi-periodic Weierstrass zeta function $\zeta(z)$, and the multiplicative doubly quasi-periodic Weierstrass sigma function $\sigma(z)$ all then are generated quite simply by a flexible variety of sequences typified by [but by no means limited to[39(Sec. 18.10)]]:

$$e_1 = P(\omega_1) = (\pi/2\omega_1)^2[\theta_3^4(0)+\theta_4^4(0)]/3, \tag{128}$$

$$e_2 = P(\omega_2) = -(\pi/2\omega_1)^2[\theta_2^4(0)+\theta_3^4(0)]/3, \tag{129}$$

$$e_3 = P(\omega_3) = (\pi/2\omega_1)^2[\theta_2^4(0)-\theta_4^4(0)]/3, \tag{130}$$

$$\eta_1 = \zeta(\omega_1) = -(\pi/2)^2[\theta_1'''(0)/3\theta_1'(0)]/\omega_1, \tag{131}$$

$$\eta_2 = \zeta(\omega_2) = \eta_1\hat{\tau}-i\pi/2\omega_1, \tag{132}$$

$$\eta_3 = \zeta(\omega_3) = -\eta_1-\eta_2, \tag{133}$$

$$P(z) = e_1+[(\pi/2\omega_1)\theta_2(\pi z/2\omega_1)\theta_1'(0)/\theta_1(\pi z/2\omega_1)\theta_2(0)]^2, \tag{134}$$

$$P(z) = e_2+[(\pi/2\omega_1)\theta_4(\pi z/2\omega_1)\theta_1'(0)/\theta_1(\pi z/2\omega_1)\theta_4(0)]^2, \tag{135}$$

$$P(z) = e_3+[(\pi/2\omega_1)\theta_3(\pi z/2\omega_1)\theta_1'(0)/\theta_1(\pi z/2\omega_1)\theta_3(0)]^2, \tag{136}$$

$$\zeta(z) = \eta_1 z/\omega_1+(\pi/2\omega_1)\theta_1'(\pi z/2\omega_1)/\theta_1(\pi z/2\omega_1), \tag{137}$$

$$\sigma(z) = (2\omega_1/\pi)[\theta_1(\pi z/2\omega_1)/\theta_1'(0)]\ \exp[\eta_1 z^2/2\omega_1]. \tag{138}$$

Logarithmic differentiation of the transparent identity,

$$P'(z) = -2(\pi/2\omega_1)^3[\theta_1'(0)]^2[\theta_2(\pi z/2\omega_1)\theta_3(\pi z/2\omega_1)\theta_4(\pi z/2\omega_1)/\theta_1^3(\pi z/2\omega_1)], \quad (139)$$

followed by cyclic half-period phase shifts, generates the quartic roots (50-53) from the symmetric sequence,

$$P''(z)/P'(z) = (\pi/2\omega_1)[\theta_2'(\pi z/2\omega_1)/\theta_2(\pi z/2\omega_1)+\theta_3'(\pi z/2\omega_1)/\theta_3(\pi z/2\omega_1)$$
$$+\theta_4'(\pi z/2\omega_1)/\theta_4(\pi z/2\omega_1)-3\theta_1'(\pi z/2\omega_1)/\theta_1(\pi z/2\omega_1)], \quad (140)$$

$$P''(z+\omega_1)/P'(z+\omega_1) = (\pi/2\omega_1)[\theta_3'(\pi z/2\omega_1)/\theta_3(\pi z/2\omega_1)+\theta_4'(\pi z/2\omega_1)/\theta_4(\pi z/2\omega_1)$$
$$+\theta_1'(\pi z/2\omega_1)/\theta_1(\pi z/2\omega_1)-3\theta_2'(\pi z/2\omega_1)/\theta_2(\pi z/2\omega_1)], \quad (141)$$

$$P''(z+\omega_2)/P'(z+\omega_2) = (\pi/2\omega_1)[\theta_1'(\pi z/2\omega_1)/\theta_1(\pi z/2\omega_1)+\theta_2'(\pi z/2\omega_1)/\theta_2(\pi z/2\omega_1)$$
$$+\theta_3'(\pi z/2\omega_1)/\theta_3(\pi z/2\omega_1)-3\theta_4'(\pi z/2\omega_1)/\theta_4(\pi z/2\omega_1)], \quad (142)$$

$$P''(z+\omega_3)/P'(z+\omega_3) = (\pi/2\omega_1)[\theta_4'(\pi z/2\omega_1)/\theta_4(\pi z/2\omega_1)+\theta_1'(\pi z/2\omega_1)/\theta_1(\pi z/2\omega_1)$$
$$+\theta_2'(\pi z/2\omega_1)/\theta_2(\pi z/2\omega_1)-3\theta_3'(\pi z/2\omega_1)/\theta_3(\pi z/2\omega_1)]. \quad (143)$$

All these numerics, it must be stressed, are based directly upon the cyclicly symmetric Weierstrass definitions (as developed most recently by Whittaker[37]), both to minimize analysis labor and to simplify digital computer loops. [The cyclicly asymmetric Jacobi definitions (as developed most recently by Southard[39]) cannot survive optimality scrutiny with respect to either analysis labor or digital computer efficiency.]

PARAMETER VARIATION

Variation of parameters [such as the parametric basis $(\underline{\Lambda}, a, b, K, \underline{\omega}_1, \underline{\omega}_2, \underline{\delta})$ spanning the preceding Vlasov-Maxwell ion-plasma solutions] to meet intricately coupled constraints [such as the cascaded horrors (35-38,50-53,110-111,113)] involves, in general, varying a subset $\underline{\alpha}$ of input parameters to obtain a subset $\underline{\beta}(\underline{\alpha})$ of output responses which satisfy a subset of constraints $\underline{\gamma}(\underline{\beta}) = 0$. Mathematically, this means inverting the cascaded vector function equation,

$$\underline{\gamma}(\underline{\beta}(\underline{\alpha})) = 0, \tag{144}$$

to obtain $\underline{\alpha}$. In the context of machine analysis, $\underline{\alpha}$ consists of some of the input data, $\underline{\beta}(\underline{\alpha})$ consists of some of the output, and $\underline{\gamma}(\underline{\beta})$ is a specified evaluation function measuring the extent to which the parameters $\underline{\alpha}$ depart from constraint requirements.

Consider looping the entire machine analysis, by varying $\underline{\alpha}$ systematically in successive passes by small input-controlled increments $\underline{\Delta}(\underline{\alpha})$ to generate a matrix difference-ratio representation of the error/value sensitivity gradient $[\underline{\Delta}(\underline{\alpha})]^{-1}\cdot[\underline{\Delta}(\underline{\gamma})]$, and then expanding the constraints (144) in the form

$$\underline{\gamma}(\underline{\beta}(\underline{\alpha})) + (d\underline{\alpha})\cdot[\underline{\Delta}(\underline{\alpha})]^{-1}\cdot[\underline{\Delta}(\underline{\gamma})] \approx 0. \tag{145}$$

Solving this set of locally linear algebraic equations, by cascaded matrix inversion, yields for the parameter changes $d\underline{\alpha}$ needed to meet constraints,

$$d\underline{\alpha} \approx - \underline{\gamma}(\underline{\beta}(\underline{\alpha}))\cdot[\underline{\Delta}(\underline{\gamma})]^{-1}\cdot[\underline{\Delta}(\underline{\alpha})]. \tag{146}$$

It is to be noted here that the parameter vector $\underline{\alpha}$ and the constraint vector $\underline{\gamma}$ must have the same dimensionality and that the intermediate

vector $\underline{\beta}$ must have dimensionality equal to or greater than that common to $\underline{\alpha}$ and $\underline{\gamma}$. Applying the parameter refinement $d\underline{\alpha}$ to the parameter basis $\underline{\alpha}$ now generates the recursive refinement algorithm for the variational logic,

$$\boxed{\underline{\alpha} \approx \underline{\alpha} - \underline{\gamma}(\underline{\beta}(\underline{\alpha}))\cdot[\underline{\Delta}(\underline{\gamma})]^{-1}\cdot[\underline{\Delta}(\underline{\alpha})].} \tag{147}$$

CONCLUSIONS

Exact expressions for nonlinear ion orbits under the influence of an axial magnetic field and an arbitrary rotary wave have been derived in terms of complex elliptic functions. As a direct byproduct of these momentum and position space expressions, the complete set of six non-trivial constants of the motion were identified. The constants permitted the Vlasov equation to be satisfied identically by a flexible phase-space distribution function of the constants. This functional form allowed Maxwell's equations to be integrated completely to obtain a nontrivial eigenvalue equation.

Exact nonlinear expressions for the phase-space distribution are obtained explicitly for two- and three-ion species in terms of six complex eigenvalues. Variation of starting values that will yield physically acceptable eigenvalues is not trivial since their determination must be evaluated over a set of multiply indexed double period modes. The evaluation of the complex elliptic functions is facilitated by the series expansions which converge rapidly. The numerical evaluation of the proper physical modes remains a difficult problem, but an exhaustive computer program has been formulated. The nature of the problem necessitates a parameter variation procedure in the complex eigenvalue hyperspace.

Several applications of these results remain to be explored. Various techniques for stimulation of ULF-VLF (0.1-10^5 Hz) radio noise in the magnetosphere and ionosphere require a nonlinear description. When specific experiments are contemplated, a thorough examination of the adjacent parameter domain would be advisable using the foregoing nonlinear solutions in order to determine the optimum natural conditions for good diagnostic signatures.

Fusion research using dense plasmas may also be served in various ways. The existing formalism might be useful for optimization of conditions for electromagnetic pumping of plasma energy density. Radiation loss rates for simple geometries can be evaluated at their nonlinear limit. Finally, the nonlinear theory may provide improved interpretation of observed radio noise emissions which could then be developed as a diagnostic tool.

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